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Deformations of Active Flexible Rods with Embedded Line Actuators

by

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Abstract

The 3-D formulation of flexible rods with embedded line actuators is presented in this paper. Both the rod and the line actuator are assumed to be initially curved and relatively positioned in an arbitrary way. The deformed configurations of the rod and actuator are connected by assuming that the principal planes of the rod remain plane and inextensible in the deformed configuration. The resultant forces and moments in the rod and actuator are found by solving the equilibrium equations for both the rod and actuator and applying continuity of tractions on the rod-actuator interface. The formulation is reduced to 2-D and applied to the case of a Shape Memory Alloy (SMA) fiber embedded in a cylindrical rod. The deformed shapes of the rod under repeated thermal actuation and the resulting shape memory loss due to the development of residual stresses are evaluated. Finally, the inverse problem of finding the required actuation force and initial curvature, to acquire a predetermined deformed shape, is solved in closed form for the 2-D case.

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1.0 Introduction

The one way shape memory effect (Buehler and Wiley, 1965, McNichols and Cory, 1987, Schetky and Wu, 1991) is associated with twinning induced permanent strain recovery upon heating. Embedding of shape memory alloy (SMA) fibers in slender flexible bodies results in shape changes of the host medium, whenever shape recovery of the SMA takes place. There are two methods of embedding SMA fibers in flexible structures (Chaudhry and Rogers, 1991). The prestrained SMA fibers are either inserted in a sleeve and then clamped at both ends, or they are continuously bonded with the flexible structure. In the first case the result is a point follower force applied at the ends, while in the second case a distributed actuation force is applied from the fiber to the flexible structure. This paper deals with modeling of the deformed shapes of flexible rods with continuously embedded line actuators.

The shape change of a cylindrical rod with a single off-axis embedded SMA fiber has been modelled in a recent paper by Lagoudas and Tadjbakhsh (1992). The distributed axial compressive force and bending moment due to phase transformation in the SMA fiber was evaluated using an approximate shear-lag model and the deformed shape of the rod was found by numerically solving the equations of equilibrium for the rod. In this paper we present the generalization of the theory to a flexible rod with a line actuator embedded in an arbitrary location with respect to the centroidal axis of the rod. The equations of equilibrium of the rod and actuator are derived in a systematic way for rods undergoing large 3-D deformations (bending, twist and extension). An example is presented for the case of a SMA fiber actuator undergoing repeated phase transformations between the martensitic and austenitic phases, for which the shape memory loss and the reduction in the actuation force are evaluated.

2.0 Kinematics of Flexible Rods with Embedded Line Actuators

2.1 Reference Configuration of Rod and Line Actuator

A rod is described in the reference configuration by the centroidal line, C, that connects the centroids of its cross sections and by the reference body orthonormal triad $\underset{\sim}{E}_i, i = 1, 2, 3$, where $\underset{\sim}{E}_3$ is the unit tangent vector to C and $\underset{\sim}{E}_1, \underset{\sim}{E}_2$ are aligned with the principal axes of the cross-section (Fig. 1). The actuator is geometrically characterized by its center line, i.e. the tangent vector to it. We will assume that the line actuator has no bending resistance and as a result there is no need to select a body frame dictated by the principal axes of its cross-sections. As it will become obvious later on, a natural choice for a body frame for the actuator is the orthonormal triad formed by the tangent vector to its center line, A, the principal normal and the binormal to A (trihedral basis).

The position vector $\underset{\sim}{X}$ of the line actuator in the reference configuration, with respect to a fixed Cartesian coordinate system, is given in terms of the position vector $\bar{\underset{\sim}{X}}$ of the centroidal line C of the rod by

$$\underset{\sim}{X} = \bar{\underset{\sim}{X}} + \underset{\sim}{D} \qquad \text{(EQ 1)}$$

where $\underset{\sim}{D} = d_\alpha \underset{\sim}{E}_\alpha = d_1 \underset{\sim}{E}_1 + d_2 \underset{\sim}{E}_2$ is the position vector of the actuator relative to the centroidal line C, as shown in Fig. 1. If $\underset{\sim}{n}_i, i = 1, 2, 3$ is the orthonormal triad of the fixed coordinate system, the reference body triad is expressed in terms of the direction cosines L_{ij} by $\underset{\sim}{E}_i = L_{ji}(S)\underset{\sim}{n}_j$, and eq. (1) reduces to

$$\underset{\sim}{X} = (\bar{X}_i(S) + d_\alpha(S) L_{i\alpha}(S))\, \underset{\sim}{n}_i \qquad \text{(EQ 2)}$$

where S denotes the arc length along C. The relationship between the arc length Z along the actuator A and arc length S along C is given by

$$1 = \frac{d\underset{\sim}{X}}{dZ} \cdot \frac{d\underset{\sim}{X}}{dZ} = \left(\frac{dZ}{dS}\right)^{-2} \left(1 + 2\underset{\sim}{E}_3 \cdot \frac{d\underset{\sim}{D}}{dS} + \frac{d\underset{\sim}{D}}{dS} \cdot \frac{d\underset{\sim}{D}}{dS}\right) \qquad \text{(EQ 3)}$$

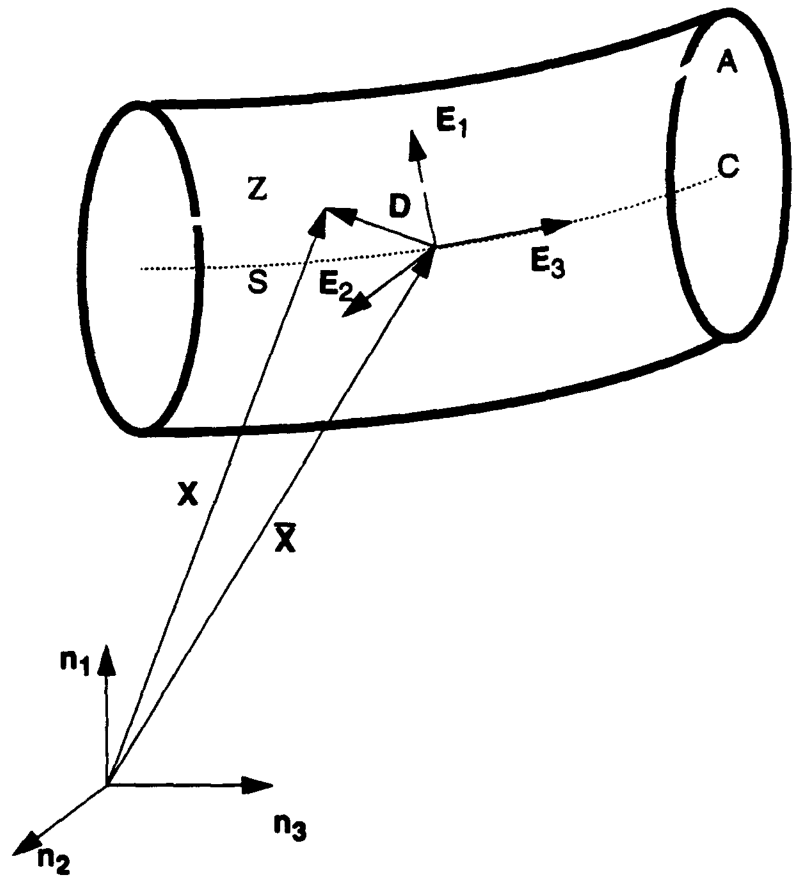
The above equation reduces to

$$\frac{dZ}{dS} = \sqrt{(1 - d_1 K_2 + d_2 K_1)^2 + \left(\frac{dd_1}{dS} - d_2 K_3\right)^2 + \left(\frac{dd_2}{dS} + d_1 K_3\right)^2} \qquad \text{(EQ 4)}$$

where the curvatures, K_i, i=1,2,3, of the reference shape of the rod are defined by

$$\frac{d\underset{\sim}{E}_i}{dS} = \varepsilon_{kji} K_j \underset{\sim}{E}_k \qquad \text{(EQ 5)}$$

FIGURE 1 Reference Configuration of Rod and Line Actuator



2.2 Deformed Configuration of Rod and Line Actuator

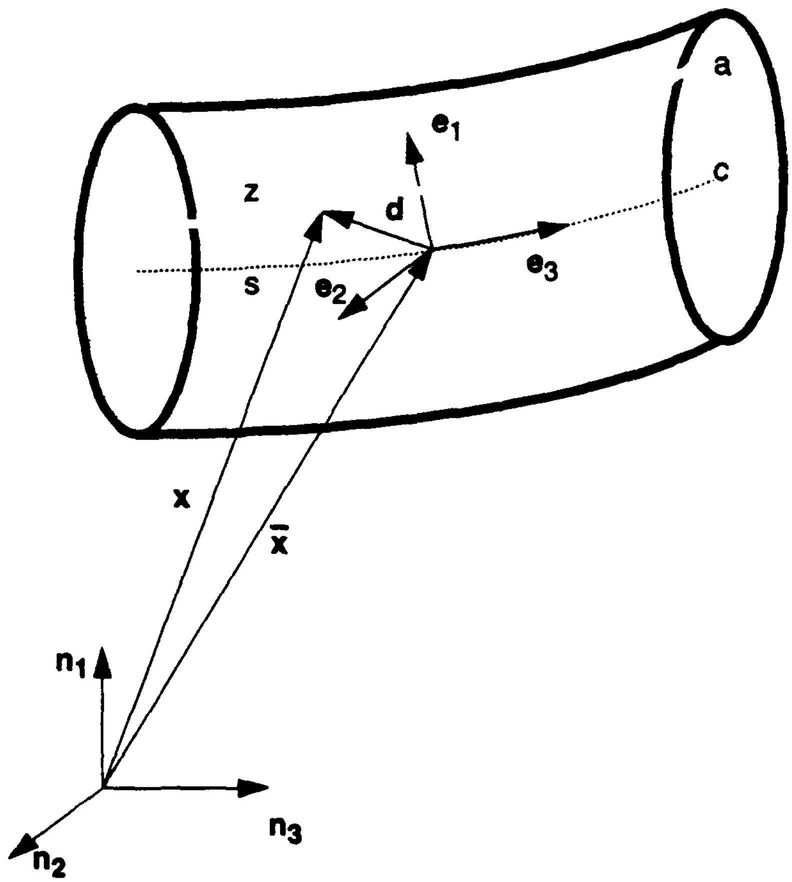
Assuming that in the deformed configuration the relative position of the line actuator with respect to the body frame remains the same, the position vector of the deformed shape of the actuator is given by

$$\underset{\sim}{x} = \underset{\sim}{\bar{x}} + \underset{\sim}{d} \qquad \text{(EQ 6)}$$

$$\underset{\sim}{x} = (\bar{x}_i(S) + d_\alpha(S) l_{i\alpha}(S))\, \underset{\sim}{n}_i \qquad \text{(EQ 7)}$$

where $\underset{\sim}{\bar{x}}$ is the position vector of the deformed centroidal line of the rod, c, and the relative position of the actuator is given by $\underset{\sim}{d} = d_\alpha \underset{\sim}{e}_\alpha$ (Fig. 2). The deformed body triad, $\underset{\sim}{e}_i, i = 1, 2, 3$, is expressed in terms of direction cosines l_{ij} by $\underset{\sim}{e}_i = l_{ji}(S)\underset{\sim}{n}_j$.

FIGURE 2 Deformed Configuration of Rod and Line Actuator



The deformed arc length z along the actuator is found in a procedure similar to the one used for Z, with the only difference that the elongation of the centroidal line c has to be taken into consideration, i.e.

$$1 = \frac{d\underline{x}}{dz} \cdot \frac{d\underline{x}}{dz} = \left(\frac{dz}{dS}\right)^{-2} \left((1+e)^2 + 2(1+e)\,\underline{E}_3 \cdot \frac{d\hat{\underline{d}}}{dS} + \frac{d\hat{\underline{d}}}{dS} \cdot \frac{d\hat{\underline{d}}}{dS} \right) \quad \text{(EQ 8)}$$

where the strain e is defined by

$$\frac{d\bar{\underline{x}}}{dS} = (1+e)\,\underline{e}_3 \quad \text{(EQ 9)}$$

Eq. (8) finally reduces to

$$\frac{dz}{dS} = \sqrt{(1+e-d_1k_2+d_2k_1)^2 + \left(\frac{dd_1}{dS} - d_2k_3\right)^2 + \left(\frac{dd_2}{dS} + d_1k_3\right)^2} \qquad \text{(EQ 10)}$$

with the curvatures, k_i, i=1,2,3, of the deformed shape of the rod defined by

$$\frac{d\underline{e}_i}{dS} = \varepsilon_{kji}k_j\underline{e}_k \qquad \text{(EQ 11)}$$

3.0 Equations of Equilibrium for Flexible Rods with Embedded Line Actuators

3.1 Equilibrium of Line Actuator

The vector form of the equations of equilibrium for the line actuator is

$$\frac{d\underline{F}^a}{dZ} + \underline{f}^a = 0 \qquad \text{(EQ 12)}$$

The internal force $\underline{F}^a$ in the actuator is tangential to its center line, i.e.

$$\underline{F}^a = F^a\underline{t} \qquad \text{(EQ 13)}$$

where the tangent unit vector, $\underline{t}$, has the evaluation

$$\underline{t} \equiv \frac{d\underline{x}}{dz} = \left(\frac{dz}{dS}\right)^{-1}\left(\left(\frac{dd_1}{dS} - d_2k_3\right)\underline{e}_1 + \left(\frac{dd_2}{dS} + d_1k_3\right)\underline{e}_2 + (1+e-d_1k_2+d_2k_1)\underline{e}_3\right) \qquad \text{(EQ 14)}$$

The distributed force, $\underline{f}^a$, is applied from the rod to the actuator and is measured per unit undeformed length of the actuator. If the trihedral basis is used to resolve the vector form of the equations of equilibrium into components, the following set of two equations obtains (equilibrium in the osculating plane)

$$\frac{dF^a}{dZ} + f_t^a = 0 \qquad \text{(EQ 15)}$$

$$F^a k + f_n^a = 0 \qquad \text{(EQ 16)}$$

The actuator curvature, k, per unit undeformed length in eq. (16) is defined by

$$k = \sqrt{\frac{d\underset{\sim}{t}}{dZ} \cdot \frac{d\underset{\sim}{t}}{dZ}} \quad \text{(EQ 17)}$$

The distributed force has in general the following resolution in the osculating plane

$$\underset{\sim}{f}^a = f_t^a \underset{\sim}{t} + f_n^a \underset{\sim}{n} \quad \text{(EQ 18)}$$

where the principal normal vector is defined by (Sokolnikoff and Redheffer, 1966)

$$\underset{\sim}{n} \equiv \frac{1}{k}\frac{d\underset{\sim}{t}}{dZ} = \frac{1}{k}\frac{d\underset{\sim}{t}}{dS}\left(\frac{dZ}{dS}\right)^{-1} \quad \text{(EQ 19)}$$

3.2 Equilibrium of Rod

The vector form of the equations of equilibrium of a flexible rod acted upon by a distributed force, $\underset{\sim}{f}$, acting at a distance $\underset{\sim}{d}$ from the centoid, is (Love, 1944, Tadjbakhsh and Lagoudas, 1992)

$$\frac{d\underset{\sim}{F}}{dS} + \underset{\sim}{f} = 0 \quad \text{(EQ 20)}$$

$$\frac{d\underset{\sim}{M}}{dS} + \frac{d\underset{\sim}{x}}{dS} \times \underset{\sim}{F} + \underset{\sim}{d} \times \underset{\sim}{f} = 0 \quad \text{(EQ 21)}$$

If we resolve the forces and moments in the body coordinate system, we have the following set of six equations

$$\frac{dF_i}{dS} + \varepsilon_{ijk} k_j F_k + f_i = 0 \quad \text{(EQ 22)}$$

$$\frac{dM_i}{dS} + \varepsilon_{ijk} k_j M_k - (1+e)\,\varepsilon_{ij3} F_j + \varepsilon_{i\alpha k} d_\alpha f_k = 0 \quad \text{(EQ 23)}$$

The explicit form of the equations of equilibrium is also given here for later reference

$$\frac{dF_1}{dS} + k_2 F_3 - k_3 F_2 + f_1 = 0 \quad \text{(EQ 24)}$$

$$\frac{dF_2}{dS} + k_3 F_1 - k_1 F_3 + f_2 = 0 \quad \text{(EQ 25)}$$

$$\frac{dF_3}{dS} + k_1 F_2 - k_2 F_1 + f_3 = 0 \quad \text{(EQ 26)}$$

$$\frac{dM_1}{dS} + k_2 M_3 - k_3 M_2 - (1+e) F_2 + d_2 f_3 = 0 \quad \text{(EQ 27)}$$

$$\frac{dM_2}{dS} + k_3 M_1 - k_1 M_3 + (1+e) F_1 - d_1 f_3 = 0 \quad \text{(EQ 28)}$$

$$\frac{dM_3}{dS} + k_1 M_2 - k_2 M_1 + d_1 f_2 - d_2 f_1 = 0 \quad \text{(EQ 29)}$$

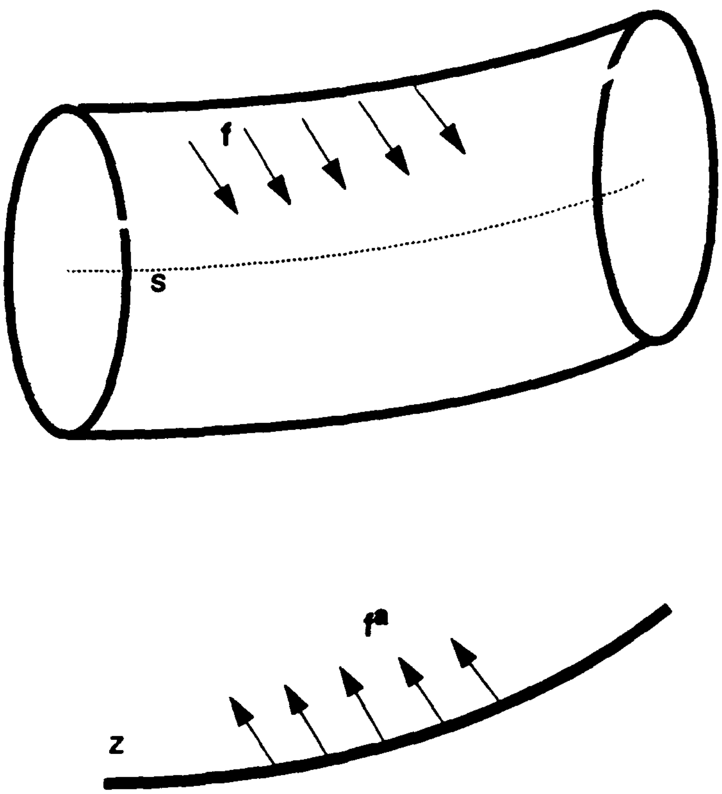
In the above equations $F_1(s)$, $F_2(s)$ and $F_3(s)$ are the two shear forces and the axial force in the rod, respectively; k_1, k_2, k_3, are the two curvatures and the twist of the deformed shape of the rod per unit undeformed length, respectively; K_1, K_2, K_3, are the curvatures and the twist of the undeformed shape of the rod in the reference configuration, respectively. The moments of inertia of a cross section of the rod, with respect to the principal axes, are denoted by I_{11} and I_{22}, while the polar moment of inertia of the cross section is denoted by I_{33}.

The distributed force, $\underset{\sim}{f}$, applied on the rod from the actuator is connected through Newton's third law to the distributed force, $\underset{\sim}{f}^a$, applied from the rod to the actuator (Fig. 3) by

$$\underset{\sim}{f} = -\frac{dZ}{dS}\underset{\sim}{f}^a \quad \text{(EQ 30)}$$

Notice that to be able to find the components of $\underset{\sim}{f}$ in terms of the components of $\underset{\sim}{f}^a$, the distributed force $\underset{\sim}{f}^a = f_t^a \underset{\sim}{t} + f_n^a \underset{\sim}{n}$ must be resolved in the body coordinate system.

FIGURE 3 Distributed Forces Applied to the Rod and Line Actuator



If linear constitutive relations are used the three bending moments and the axial force are related to the curvatures and extension by the following equations (Tadjbakhsh and Lagoudas, 1992)

$$M_1 = EI_{11}(k_1 - K_1) \qquad \text{(EQ 31)}$$

$$M_2 = EI_{22}(k_2 - K_2) \qquad \text{(EQ 32)}$$

$$M_3 = GI_{33}(k_3 - K_3) \qquad \text{(EQ 33)}$$

$$F_3 = AEe + E(I_{11}k_1(k_{11} - K_{11}) + I_{22}k_2(k_2 - K_2)) \qquad \text{(EQ 34)}$$

E and G are both material constants relating the difference in the curvatures with the corresponding moments and can be approximately taken equal with the Young's modulus and shear modulus of the material of the rod. Eqs. (24-29) have to be solved for $F_1, F_2, e, k_1, k_2, k_3$. If the rod is inextensible, i.e., e=0, eq. (34) is not valid any longer and F_3 becomes an unknown variable replacing e in eqs. (24-29).

The connection between the curvatures and the body angles of rotation $\varphi_1\underline{e}_1, \varphi_2\underline{e}_2, \varphi_3\underline{e}_3$ (Kane, Likins and Levinson, 1983), which bring the body triad from the fixed basis $\underline{n}_1, \underline{n}_2, \underline{n}_3$ to the deformed configuration $\underline{e}_1, \underline{e}_2, \underline{e}_3$ is explicitly given by (Tadjbakhsh and Lagoudas, 1992)

$$k_1 = \frac{d\varphi_1}{ds}\cos\varphi_2\cos\varphi_3 + \frac{d\varphi_2}{ds}\sin\varphi_3 \qquad \text{(EQ 35)}$$

$$k_2 = \frac{d\varphi_2}{ds}\cos\varphi_3 - \frac{d\varphi_1}{ds}\cos\varphi_2\sin\varphi_3 \qquad \text{(EQ 36)}$$

$$k_3 = \frac{d\varphi_3}{ds} + \frac{d\varphi_1}{ds}\sin\varphi_2 \qquad \text{(EQ 37)}$$

Similarly the connection between the reference curvatures and the body angles of rotation $\Phi_1\underline{E}_1, \Phi_2\underline{E}_2, \Phi_3\underline{E}_3$, which bring the body triad from the fixed basis $\underline{n}_1, \underline{n}_2, \underline{n}_3$ to the reference configuration $\underline{E}_1, \underline{E}_2, \underline{E}_3$ is explicitly given by

$$K_1 = \frac{d\Phi_1}{ds}\cos\Phi_2\cos\Phi_3 + \frac{d\Phi_2}{ds}\sin\Phi_3 \qquad \text{(EQ 38)}$$

$$K_2 = \frac{d\Phi_2}{ds}\cos\Phi_3 - \frac{d\Phi_1}{ds}\cos\Phi_2\sin\Phi_3 \qquad \text{(EQ 39)}$$

$$K_3 = \frac{d\Phi_3}{ds} + \frac{d\Phi_1}{ds}\sin\Phi_2 \qquad \text{(EQ 40)}$$

The position vector of the centroidal line of the rod is given in both the deformed and the reference configurations, respectively, by

$$\bar{x}_1 = \int_0^S (1+e)\, l_{31}\, d\bar{S} + \underline{\bar{x}}_1(0) = \int_0^S (1+e)\, \sin\varphi_2 d\bar{S} + \bar{x}_1(0) \qquad \text{(EQ 41)}$$

$$\bar{x}_2 = \int_0^S (1+e)\, l_{32} d\bar{S} + \bar{x}_2(0) = -\int_0^S (1+e)\, \sin\varphi_1 \cos\varphi_2 d\bar{S} + \bar{x}_2(0) \quad \text{(EQ 42)}$$

$$\bar{x}_3 = \int_0^S (1+e)\, l_{33} d\bar{S} + \bar{x}_3(0) = \int_0^S (1+e)\, \cos\varphi_1 \cos\varphi_2 d\bar{S} + \bar{x}_3(0) \quad \text{(EQ 43)}$$

$$\bar{X}_1 = \int_0^S L_{31} d\bar{S} + \bar{X}_1(0) = \int_0^S \sin\Phi_2 d\bar{S} + \bar{X}_1(0) \quad \text{(EQ 44)}$$

$$\bar{X}_2 = \int_0^S L_{32} d\bar{S} + \bar{X}_2(0) = -\int_0^S \sin\Phi_1 \cos\Phi_2 d\bar{S} + \bar{X}_2(0) \quad \text{(EQ 45)}$$

$$\bar{X}_3 = \int_0^S L_{33} d\bar{S} + \bar{X}_3(0) = \int_0^S \cos\Phi_1 \cos\Phi_2 d\bar{S} + \bar{X}_3(0) \quad \text{(EQ 46)}$$

In the case of an inextensible rod (e=0), eqs. (41-43) satisfy the constrain $\frac{d\bar{x}}{dS} \cdot \frac{d\bar{x}}{dS} = 1$.

4.0 2-D Bending of a Rod by an Off-Axis Line Actuator

For 2-D bending about e_2 we assume that $K_1 = K_3 = 0$ $\varphi_1 = \varphi_3 = 0$. The position vector of the deformed configuration of the actuator is given by

$$x = \bar{x} + d_1(S) e_1 \quad \text{(EQ 47)}$$

and the tangent vector to its center line has the evaluation

$$t = \frac{\frac{dd_1}{dS} e_1 + (1 + e - d_1 k_2)\, e_3}{\sqrt{(1 + e - d_1 k_2)^2 + \left(\frac{dd_1}{dS}\right)^2}} \quad \text{(EQ 48)}$$

If we define

$$\tan\phi = \frac{\frac{dd_1}{dS}}{1 + e - d_1 k_2} \tag{EQ 49}$$

the tangent and principal normal vectors of the line actuator take the form

$$\underset{\sim}{t} = \sin\phi \underset{\sim}{e}_1 + \cos\phi \underset{\sim}{e}_3 \tag{EQ 50}$$

$$\underset{\sim}{n} = \cos\phi \underset{\sim}{e}_1 - \sin\phi \underset{\sim}{e}_3 \tag{EQ 51}$$

The above formulation leads to the following evaluation of distributed forces and moments on the rod

$$f_1 = -(\cos(\phi) f_n^a + \sin(\phi) f_t^a) \frac{dZ}{dS} \tag{EQ 52}$$

$$f_3 = -(\cos(\phi) f_t^a - \sin(\phi) f_n^a) \frac{dZ}{dS} \tag{EQ 53}$$

$$m_2 \equiv \varepsilon_{2jk} d_j f_k = d_1 (\cos(\phi) f_t^a - \sin(\phi) f_n^a) \frac{dZ}{dS} \tag{EQ 54}$$

where from eq. (4) we have

$$\frac{dZ}{dS} = \sqrt{(1 - d_1 K_2)^2 + \frac{dd_1}{dS}^2} \tag{EQ 55}$$

4.1 Line Actuator Placed Parallel to the Centroidal Line of a Straight Rod

The results presented in the previous section simplify in the case of an actuator embedded parallel to the centroidal line of an initially straight rod. Since $\frac{dd_1}{dS} = 0$, eqs. (50,51) simplify to

$$\underset{\sim}{t} = \underset{\sim}{e}_3 \tag{EQ 56}$$

$$\underset{\sim}{n} = \underset{\sim}{e}_1 \qquad \text{(EQ 57)}$$

and the curvature of the actuator from eq. (19) becomes

$$k = k_2 \qquad \text{(EQ 58)}$$

Finally, eqs. (52-54) reduce to (note that ϕ=0 and dZ/dS=1)

$$f_1 = -f_n^a \qquad \text{(EQ 59)}$$

$$f_3 = -f_t^a \qquad \text{(EQ 60)}$$

$$m_2 = -d_1 f_3 \qquad \text{(EQ 61)}$$

If the above simplifications are taken into account, the equilibrium equations for the actuator (15, 16) take the form

$$\frac{dF^a}{dS} = f_3 \qquad \text{(EQ 62)}$$

$$F^a k_2 = f_1 \qquad \text{(EQ 63)}$$

while the equations of equilibrium for the rod (24 - 29) reduce to

$$\frac{dF_1}{dS} + k_2 F_3 + k_2 F^a = 0 \qquad \text{(EQ 64)}$$

$$\frac{dF_3}{dS} - k_2 F_1 + \frac{dF^a}{dS} = 0 \qquad \text{(EQ 65)}$$

$$\frac{dM_2}{dS} + (1+e) F_1 - d_1 \frac{dF^a}{dS} = 0 \qquad \text{(EQ 66)}$$

Eqs. (64-66) are the same with the ones reported in Lagoudas and Tadjbakhsh (1992), except for the term $k_2 F^a$, which is not included there. This term becomes important for large deflections with large bending curvature, while it can be neglected for cases with small resulting bending curvature. The kinematic relationships for the deformed shape, i.e., eqs. (35-37, 41-43), likewise reduce to

$$k_2 = \frac{d\varphi_2}{dS} \tag{EQ 67}$$

$$\bar{x}_1 = \int_0^S (1+e)\sin\varphi_2 d\bar{S} + \bar{x}_1(0) \tag{EQ 68}$$

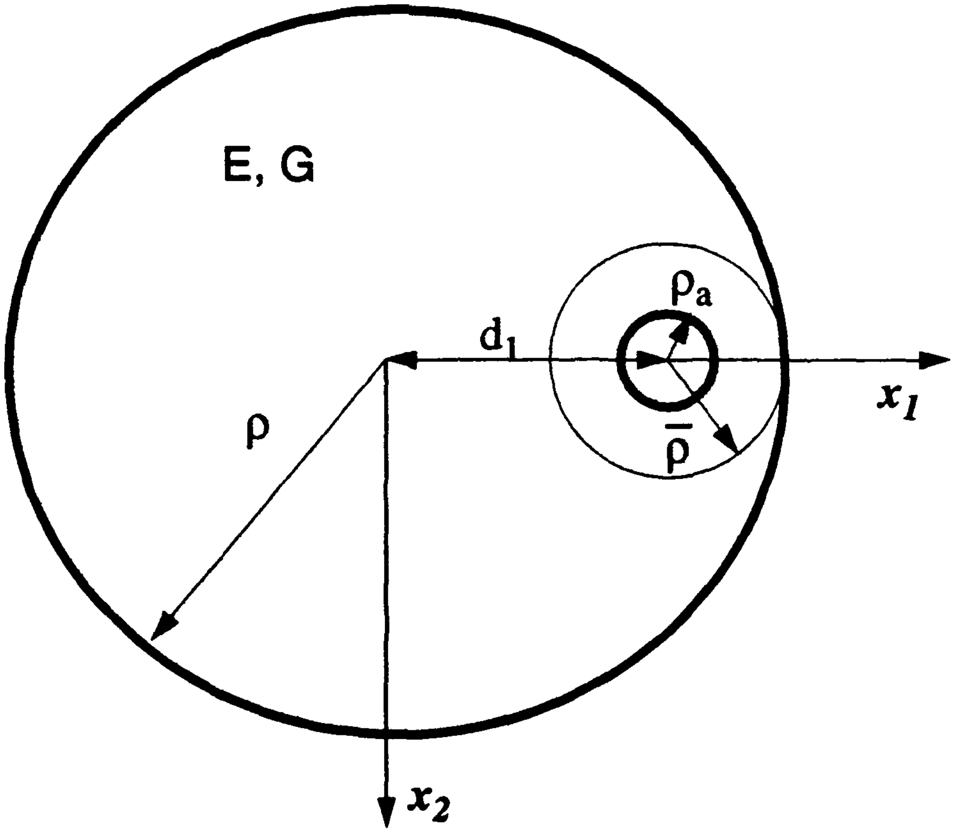
$$\bar{x}_3 = \int_0^S (1+e)\cos\varphi_2 d\bar{S} + \bar{x}_3(0) \tag{EQ 69}$$

The results derived in this section will be used to model the response of a rod to an embedded SMA fiber upon attempted shape recovery of the SMA fiber due to the martensitic-austenitic phase transformation.

5.0 2-D Bending of a Rod Induced by an Embedded Shape Memory Alloy Fiber

Let a SMA fiber be prestrained while in the martensitic phase and embedded in a flexible rod and bonded with the rod material in a temperature below M_f (Schetky and Wu, 1991, Pfaeffle et al., 1992). Upon heating above A_f, the martensite will transform to austenite and the initial inelastic strain would be recovered, were it not for the constraint provided by the rod. For a flexible rod with an off-axis embedded SMA fiber, such as the one shown in Fig. 4, the evaluation of the actuation force based on a shear-lag model has been given by Lagoudas and Tadjbakhsh (1992). For completeness we summarize here the key results.

FIGURE 4 A SMA Fiber Embedded Off-axis in a Flexible Rod



The shear-lag equations of equilibrium are

$$\frac{d\sigma^a}{dS} + \frac{2}{\rho_a}\sigma_{r3}\big|_{r=\rho_a} = 0 \qquad 0 \le S \le L \tag{EQ 70}$$

$$\frac{\partial\sigma_{r3}}{\partial r} + \frac{1}{r}\sigma_{r3} = 0 \qquad \rho_a < r < \rho - d_1 \tag{EQ 71}$$

$$\frac{d\sigma}{dS} - \frac{2}{\rho - d_1}\sigma_{r3}\big|_{r=\rho-d_1} = 0 \qquad 0 \le S \le L \tag{EQ 72}$$

Eq. (70) gives the force equilibrium in the SMA fiber in the axial direction, eq. (71) is the force equilibrium in the shear layer of thickness (ρ-d_1) in the radial direction and eq. (72) is the force equilibrium in the surrounding rod material in the axial direction. The constitutive equations for the stresses in the fiber, σ^a, shear layer, σ_{r3}, and rod material, σ, are respectively given by

$$\sigma^a = E_a\left(\frac{dw_a}{dS} - \varepsilon^{sma}\right) \quad \text{(EQ 73)}$$

$$\sigma_{r3} = G\frac{\partial w}{\partial r} \quad \text{(EQ 74)}$$

$$\sigma = E\frac{d}{dS}w\big|_{r = \rho - d_1} \quad \text{(EQ 75)}$$

where $w_a(S)$ is the axial displacement of the SMA fiber and $w(r,S)$ is the axial displacement of the surrounding rod material. E and G denote the moduli of the rod material, while E_a is the Young's modulus of the fiber in the austenitic phase.

Combining eqs. (70-75), together with the condition of overall equilibrium of forces in the axial direction, the final equation for the force in the SMA fiber, $F^a = \sigma^a\pi\rho_a^2$, is found to be (Lagoudas and Tadjbakhsh, 1992)

$$\frac{d^2F^a}{dS^2} - \alpha^2F^a = -F_a^{sma}\alpha^2 \quad \text{(EQ 76)}$$

where

$$F_a^{sma} = -\frac{\pi\rho_a^2E\varepsilon^{sma}}{\frac{E}{E_a} + \frac{\rho_a^2}{\rho^2 - \rho_a^2}} \quad \text{(EQ 77)}$$

$$\alpha^2 = \frac{2G}{E\rho_a^2 ln((\rho - d_1)/\rho_a)}\left(\frac{E}{E_a} + \frac{\rho_a^2}{\rho^2 - \rho_a^2}\right) \quad \text{(EQ 78)}$$

We will refer to F^a as the actuation force, because it is the driving force in eqs. (64-66) and gives rise to the distributed force applied to the rod through eqs. (62, 63). As eq. (73) suggests, ε^{sma} is the stress free recoverable transformation strain when the SMA fiber fully transforms into the austenitic phase. Imposing boundary conditions F^a=0 at S=0 and S=L, the following solution obtains for the actuation force

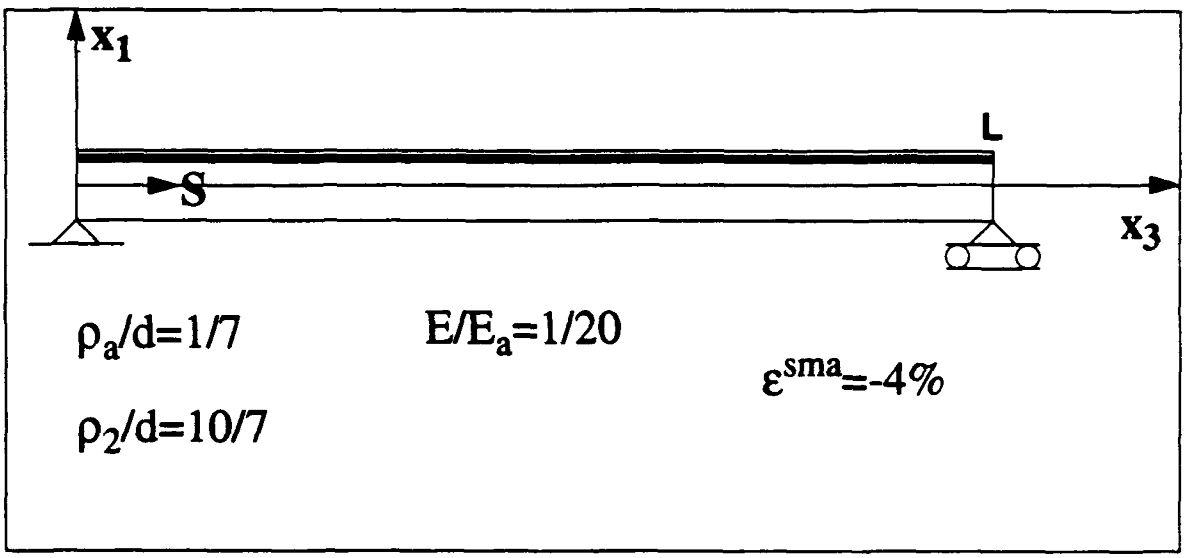
$$F^a(S) = F_a^{sma}\left(1 - \frac{e^{-\alpha S} + e^{-\alpha(L-S)}}{1 + e^{-\alpha L}}\right) \quad \text{(EQ 79)}$$

If the above evaluation of the actuation force is substituted into eqs. (64-66), these equations can be solved for F_1, e, k_2, and the deformed shape can be evaluated from eqs. (67-69). We have solved equations (64-66) numerically using the shooting method, in a similar way as described by Lagoudas and Tadjbakhsh (1992) for various boundary conditions, geometric and material parameters.

5.1 Simply Supported Rod with an Off-Axis Embedded SMA Fiber

As an illustrative example, and to present a method for finding the deformed shape of a rod with an embedded SMA fiber undergoing repeated phase transformations, we report the case of a simply supported rod, shown in Fig. 5, with an off-axis embedded SMA fiber.

FIGURE 5 A Simply Supported Rod with an Embedded SMA Fiber



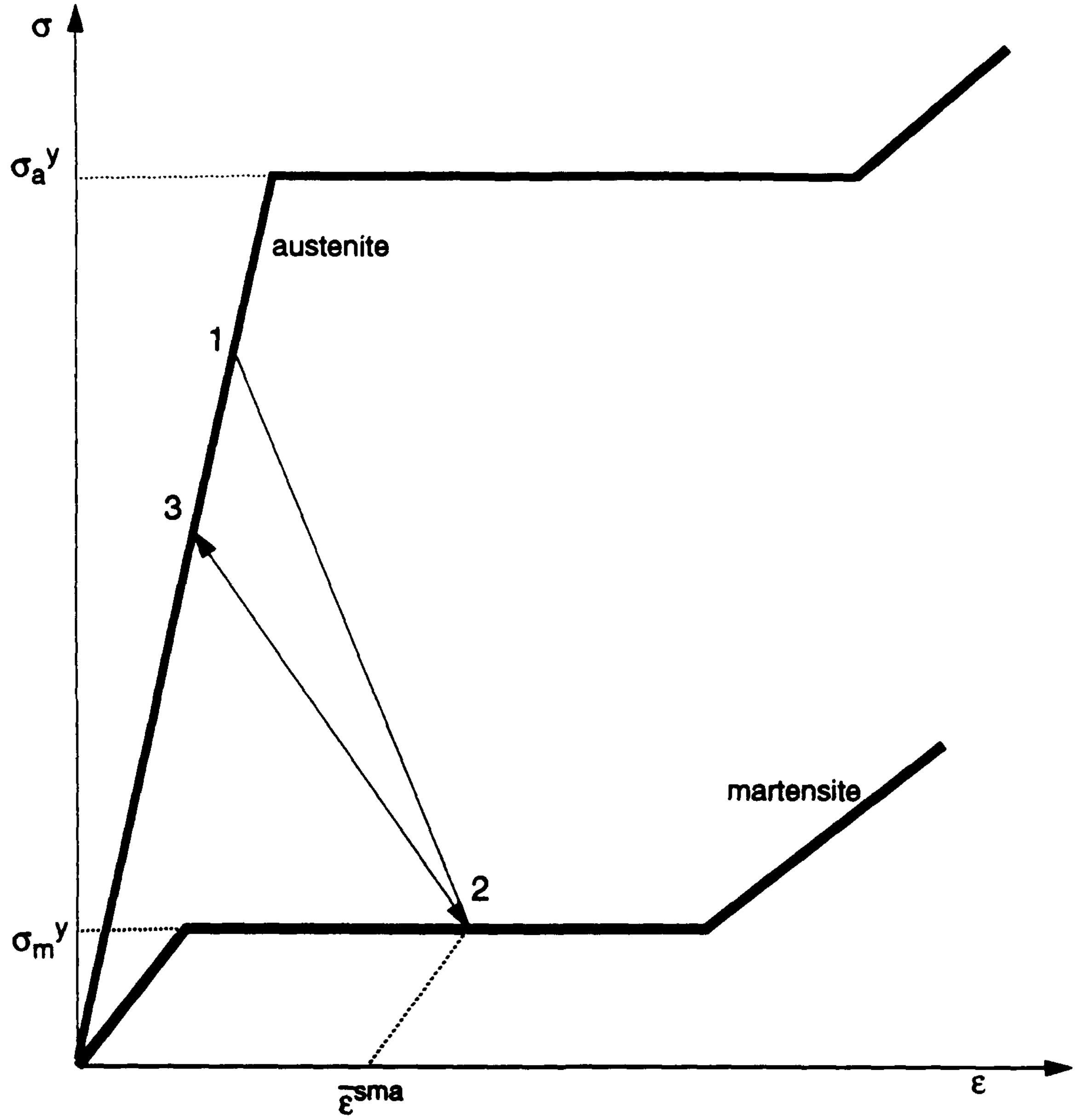
The boundary conditions for the simply supported case are given by

$$x_1(0) = k_2(0) = F_3(0) = 0 \quad \text{(EQ 80)}$$

$$x_1(L) = k_2(L) = F_3(L) = 0 \tag{EQ 81}$$

The initial stress free straight configuration is assumed to occur while the SMA fiber is in the prestrained martensitic phase before any shape recovery takes place. When heated above the phase transformation temperature, A_f, the SMA fiber transforms into the austenitic phase, attempting to recover its original shape before prestraining. The deformed shape of the rod has been evaluated for material and geometric parameters shown in Fig. 5 with initial prestrain of the SMA fiber $\varepsilon^{sma} = -4\%$. We have assumed that the austenitic phase behaves linearly elastic with $E/E_a=1/20$. The stress state in the SMA fiber is indicated schematically as state 1 in Fig. 6, and the resulting deformed shape and curvature are plotted in Figs. 8 and 9, respectively. For the numerical solution of eqs. (64-66) the rod has been assumed centrally inextensible, which is a valid assumption for SMA fibers with small diameter compared to the diameter of the rod.

FIGURE 6 Constitutive Behavior of Austenitic and Martensitic Phases

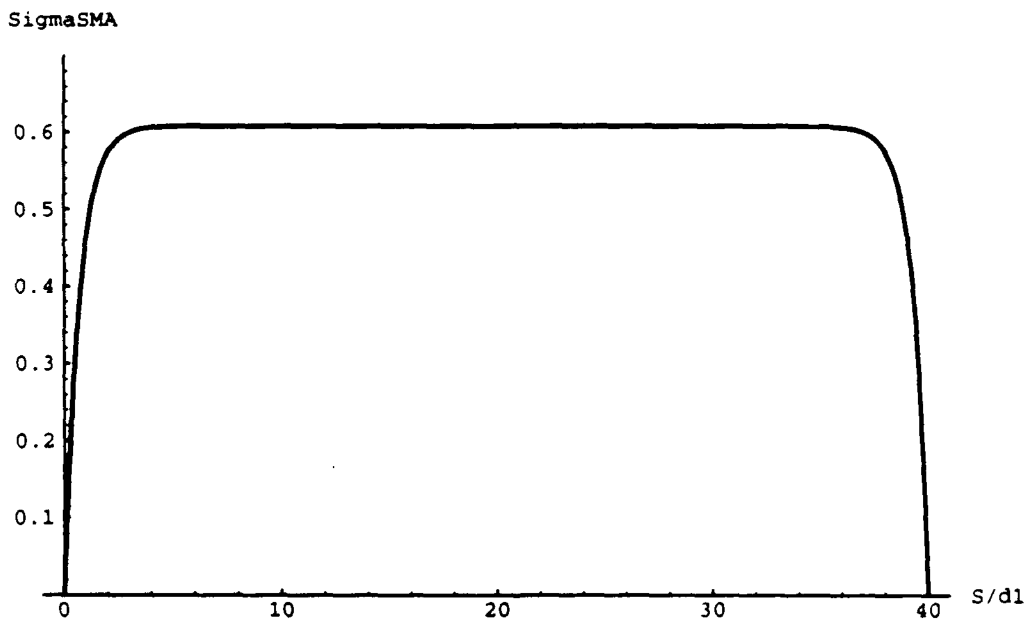


If the prestrain of the SMA fiber is large enough, the actuation force may reach the yield limit of the austenitic phase, σ_a^y, which is actually the starting point of stress induced martensitic transformation (Schetky and Wu, 1991). The condition for yielding is given by

$$F^a - E_a \pi \rho_a^2 d_1 k_2 = \pi \rho_a^2 \sigma_a^y \qquad \text{(EQ 82)}$$

The l.h.s. of the above equation, normalized by F_a^{sma}, is plotted in Fig. 7 below. The normalized value on the r.h.s. is approximately equal to the ratio $\left(\frac{\sigma_a^y}{E_a}\right)/(-\varepsilon^{sma})$. We can therefore conclude for this case from Fig. 7 that, in order not to have yielding at any point along the SMA fiber, the yield strain of the martensite must be at least 0.6 of the prestrain, or 2.4%.

FIGURE 7 Normalized stress in the SMA fiber



If yielding occurs in the SMA fiber, we assume that shear-lag model is still valid and the actuation force can be found from eq. (79) but with the replacement of eq. (77) by

$$F_a^{sma} = E_a \pi \rho_a^2 \left(\frac{\sigma_a^y}{E_a} + d_1 k_2 \left(\frac{L}{2}\right) \right) \qquad \text{(EQ 83)}$$

The above equation implies a perfectly plastic model for the austenitic phase, as shown in Fig. 6, which is an experimentally verified valid assumption, due to the formation of the compliant stress induced martensite (Sato and Tanaka, 1988). The above procedure requires an iterative solution scheme for the evaluation of the bending curvature in eq. (83).

After the rod cools down below M_f, and the SMA fiber returns to its martensitic phase, the rod does not return to the initial straight configuration, but it has some residual curvature due to residual stress development. To evaluate the actuation force in this case we assume that, after the SMA fiber transforms to the martensitic phase in a volume preserving transformation, it is stretched by the deformed rod until a new equilibrium is established. Since the martensitic phase is much more compliant and with a lower yield stress than the austenitic phase (Schetky and Wu, 1991), it yields giving rise to new prestraining. To model the constitutive behavior of the martensite we use an elastic perfectly plastic model as shown in Fig. 6, where the cool-down process is indicated by the path 1-2. The assumption of a perfectly plastic model is reasonable because the inelastic strain is mainly due to twinning stretching and to a lesser extent to dislocation motion with a very small amount of hardening (Schetky and Wu, 1991).

The actuation force, F^m, in the martensitic phase is found by the shear-lag model described by an equation similar to eq. (76) with the solution given by

$$F^m(S) = F_m^{sma}\left(1 - \frac{e^{-\beta S} + e^{-\beta(L-S)}}{1 + e^{-\beta L}}\right) \tag{EQ 84}$$

with the following evaluation for the constants

$$\beta^2 = \frac{2G}{E\rho_a^2 ln((\rho - d_1)/\rho_a)}\left(\frac{E}{E_m} + \frac{\rho_a^2}{\rho^2 - \rho_a^2}\right) \tag{EQ 85}$$

$$F_m^{sma} = E_m \pi \rho_a^2\left(\frac{\sigma_m^y}{E_m} + d_1 k_2\left(\frac{L}{2}\right)\right) \tag{EQ 86}$$

where E_m is the stiffness and $\sigma_m{}^y$ is the yield stress of the martensitic phase. The deformed shape of the rod corresponding to the martensitic phase is shown as curve 2 in Fig 8 and the bending curvature in Fig. 9. For the martensitic phase we have used E/E_m=1/6 and $\sigma_m{}^y/E_m$=0.01.

FIGURE 8 Deformed Shape of a Rod Under Cyclic Actuation of the Embedded SMA Fiber

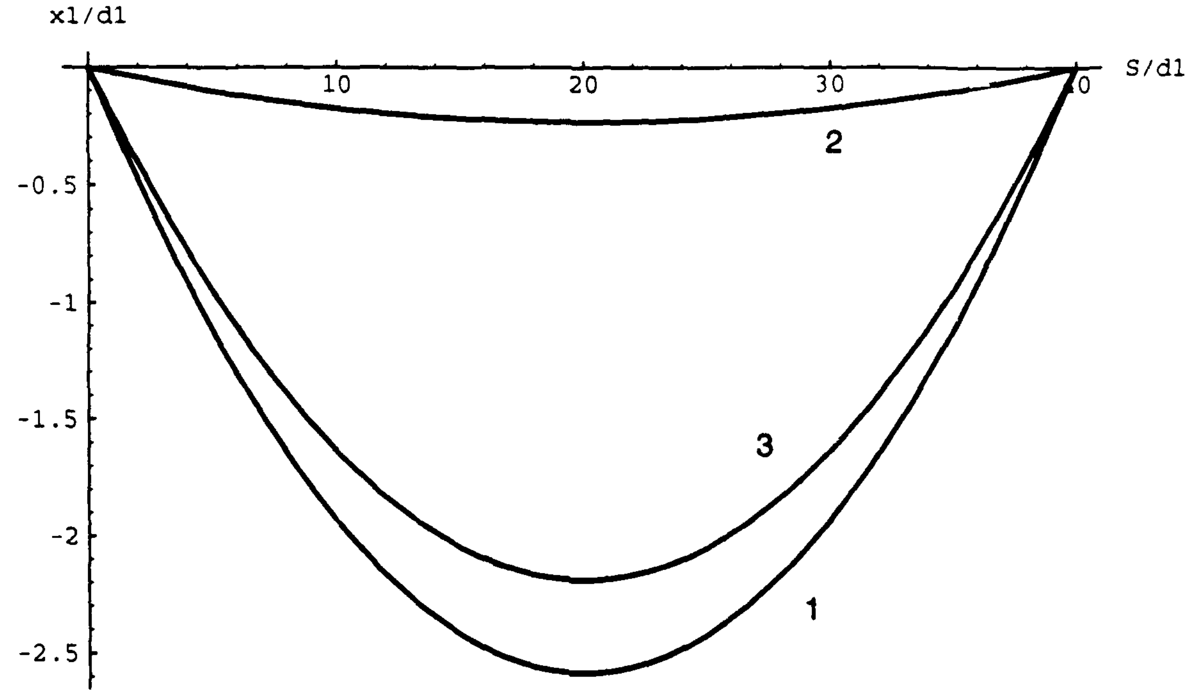
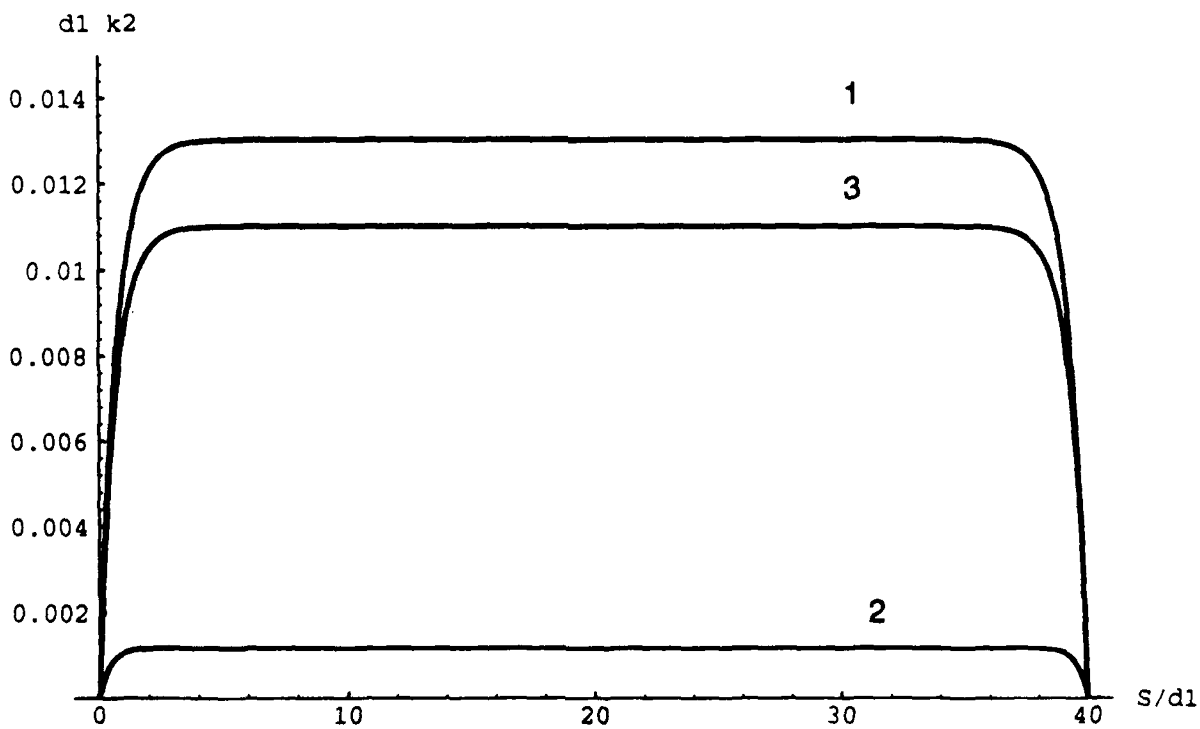


FIGURE 9 Bending Curvature of a Rod Under Cyclic Actuation of the Embedded SMA Fiber



To model subsequent cycles, the transformation strain has to be corrected to include residual stresses in the SMA fiber after transformation from austenite to martensite at the end of process 1-2. The new transformation strain, $\hat{\varepsilon}^{sma}$, is given by the initial transformation strain reduced by the residual elastic strain of the SMA fiber, i.e.,

$$\hat{\varepsilon}^{sma} = \varepsilon^{sma} - \left(\frac{F^m}{\pi\rho_a^2 E_m} - d_1 k_2\right) \tag{EQ 87}$$

Substitution from eq. (84) into the above formula results in

$$\hat{\varepsilon}^{sma}(S) = \varepsilon^{sma} + d_1 k_2(S) - \frac{F_m^{sma}}{\pi\rho_a^2 E_m}\left(1 - \frac{e^{-\beta S} + e^{-\beta(L-S)}}{1 + e^{-\beta L}}\right) \tag{EQ 88}$$

Far away from the ends, the above formula reduces to a simple one

$$\hat{\varepsilon}^{sma}\left(\frac{L}{2}\right) = \varepsilon^{sma} - \frac{\sigma_m^y}{E_m} \tag{EQ 89}$$

Utilizing eq. (88) for the shape memory loss, the deformed configuration and bending curvature of the rod is evaluated and shown as curves (3) in Figs. 8 and 9, respectively. Subsequent cycles will not influence the residual stress state and the rod will be deforming between states (2) and (3).

As a final note and to bring some connection with the composite rod theory, we rewrite eqs. (64-66) in the following form

$$\frac{dF_1}{dS} + k_2(F_3 + F^a) = 0 \tag{EQ 90}$$

$$\frac{d}{dS}(F_3 + F^a) - k_2 F_1 = 0 \tag{EQ 91}$$

$$\frac{d}{dS}(M_2 - d_1 F^a) + (1 + e) F_1 = 0 \tag{EQ 92}$$

Away from the ends of the rod and for vanishingly small diameter of the SMA fiber, we have

$$F^a \approx F_a^{sma} = -\frac{\pi\rho_a^2 E\varepsilon^{sma}}{\frac{E}{E_a} + \frac{\rho_a^2}{\rho^2 - \rho_a^2}} \approx -\pi\rho_a^2 E_a \varepsilon^{sma} \qquad \text{(EQ 93)}$$

If we consider the actuation force as an internal force for the composite rod-SMA fiber system due to the phase transformation strain, we could absorb the actuation force in the constitutive equations (31-34), i.e.,

$$\tilde{M}_2 = EI_{22}(k_2 - K_2) + d_1(\pi\rho_a^2 E_a \varepsilon^{sma}) \qquad \text{(EQ 94)}$$

$$\tilde{F}_3 = AEe + E(I_{11}k_1(k_1 - K_1) + I_{22}k_2(k_2 - K_2)) - \pi\rho_a^2 E_a \varepsilon^{sma} \qquad \text{(EQ 95)}$$

in which case the equations of equilibrium become

$$\frac{dF_1}{dS} + k_2\tilde{F}_3 = 0 \qquad \text{(EQ 96)}$$

$$\frac{d\tilde{F}_3}{dS} - k_2 F_1 = 0 \qquad \text{(EQ 97)}$$

$$\frac{d\tilde{M}_2}{dS} + (1+e)F_1 = 0 \qquad \text{(EQ 98)}$$

For small curvature and extension, equations (94-98) reduce to the classical composite beam equations (Allen and Haisler, 1985). The shear-lag model, therefore, in the case of line actuators is compatible with the composite beam theory and in addition takes into account the end effects.

6.0 Inverse Problems

Equations (64-66) can also be thought of as differential equations for the actuation force, F^a, provided that the bending curvature is known, i.e., a desirable deformed shape is specified a priori. In the case of inextensible rods, the system of equations (64-66) becomes linear in F_1, F_3, and F^a and it can easily be integrated. Before we go into the solution of the inverse problem in general, we give the solution to a special case below.

If we eliminate F_3 between eqs. (64) and (65), in the process F^a gets also eliminated and we end up with

$$\frac{d}{dS}\left(\frac{1}{k_2}\frac{dF_1}{dS}\right) + k_2 F_1 = 0 \qquad \text{(EQ 99)}$$

which has the general solution

$$F_1 = C_1 \cos\varphi_2 + C_2 \sin\varphi_2 \qquad \text{(EQ 100)}$$

where C_1 and C_2 are arbitrary constants. Substitution of F_1 into eq. (64) yields

$$F_3 = -F^a + C_1 \sin\varphi_2 - C_2 \cos\varphi_2 \qquad \text{(EQ 101)}$$

To obtain M_2 we substitute eq. (100) into eq. (66) which reduces to

$$\frac{dM_2}{dS} - d_1 \frac{dF^a}{dS} = -(1+e)\ (C_1 \cos\varphi_2 + C_2 \sin\varphi_2) \qquad \text{(EQ 102)}$$

Integration of the above equation yields

$$M_2 - d_1 F^a = -C_1 \bar{x}_3 - C_2 \bar{x}_1 + C_3 \qquad \text{(EQ 103)}$$

where C_3 is a constant of integration. Considering moment equilibrium of a segment of the rod we arrive at the following result

$$M_2(S) = d_1 F^a(S) - P_3\,(\bar{x}_1(L) - \bar{x}_1(S)) + P_1\,(\bar{x}_3(L) - \bar{x}_3(S)) + T_2 \qquad \text{(EQ 104)}$$

where $\underline{P} = P_1 \underline{n}_1 + P_3 \underline{n}_3$ is the externally applied force at S=L and $\underline{T} = T_2 \underline{n}_2$ is the applied moment at S=L. If sufficient boundary data are known eq. (104) can be utilized to correlate the actuation force with the deformed shape. In the case of a free-free end, $F_1=0$ at $S=0$ and $S=L$. Hence from eq. (100) $C_1=C_2=0$. Also $M_2=0$ and $F^a=0$ at $S=0$ and $S=L$ and from eq. (103) $C_3=0$. Eqs. (101) and (103) simplify in this case to

$$F_3 = -F^a \qquad M_2 = EI_{22}(k_2 - K_2) = d_1 F^a \tag{EQ 105}$$

The above simple solution indicates that if a desired shape is sought, the required actuation force is proportional to the difference of the bending curvatures in the deformed and reference states.

For more general boundary conditions, the following procedure can be followed for the solution of the inverse problem. If we assume $\epsilon = 0$ (centrally inextensible rod), differentiating eq. (66) twice and substituting from (64, 65) leads to

$$\frac{d}{dS}\left(\frac{d^2M_2}{dS^2}/k_2\right) + k_2\frac{dM_2}{dS} + \frac{d}{dS}\left(\frac{dm_2}{dS}/k_2\right) + k_2 m_2 = 0 \tag{EQ 106}$$

where $m_2=-d_1f_3$. If we substitute for M_2 from eq. (32) and define, $\hat{m}_2 = \frac{m_2}{EI_{22}}$, the above equation yields

$$\frac{d}{dS}\left(\frac{d^2}{dS^2}(k_2-K_2)/k_2\right) + k_2\frac{d}{dS}(k_2-K_2) + \frac{d}{dS}\left(\frac{d\hat{m}_2}{dS}/k_2\right) + k_2\hat{m}_2 = 0 \tag{EQ 107}$$

This equation is a third order nonlinear o.d.e for the bending curvature k_2. Notice that while eq. (104) was derived by integrating the equations of equilibrium (64-66), eq. (107) is obtained by differentiating them.

For the solution of the direct problem, where the distributed actuation force is given and the induced deformation is sought, eq. (107) or equivalently eqs. (64-66) are solved numerically using the shooting method (Lagoudas and Tadjbakhsh, 1992). Eq. (107) can be primarily utilized to solve in closed form two inverse problems:

1. Inverse Problem (a): Find m_2 for given k_2 and K_2 along the rod.
2. Inverse Problem (b): Find K_2 for given k_2 and m_2 along the rod.

For the first inverse problems eq. (107) is a linear second order o.d.e. for $\hat{m}_2$ with variable coefficients. The homogenous solution is given by

$$\hat{m}_2 = D_1 \sin\left(\int_0^S k_2 d\bar{S}\right) + D_2 \cos\left(\int_0^S k_2 d\bar{S}\right) \qquad \text{(EQ 108)}$$

The particular solution can be found using the method of variation of parameters. If the initial curvature is zero, in which case eq. (107) reduces to

$$\frac{d}{dS}\left(\frac{d\hat{m}_2}{dS}/k_2\right) + k_2\hat{m}_2 = -\frac{d}{dS}\left(\frac{d^2k_2}{dS}/k_2 + \frac{1}{2}k_2^2\right) \qquad \text{(EQ 109)}$$

We can write the general solution of eq. (109) in the following compact form

$$\hat{m}_2 = -\left(\int_0^S g(\bar{S})\cos\left(\theta(\bar{S}) - \theta(S)\right) d\bar{S} + D_1 \sin\theta(S) + D_2 \cos\theta(S)\right) \qquad \text{(EQ 110)}$$

where g(S) and θ(S) are given by

$$g(S) = \frac{d^2k_2}{dS} + \frac{1}{2}k_2^3 \qquad \text{(EQ 111)}$$

(EQ 112)

$$\theta(S) = \int_0^S k_2 d\bar{S} \qquad \text{(EQ 113)}$$

The two constants D_1 and D_2 are found by satisfying appropriate boundary conditions. As an example, for a simply supported rod, -L<S<L, eq. (110) reduces to

$$\hat{m}_2 = \left(\int_0^L g(\bar{S})\cos\left(\theta(\bar{S}) - \theta(L)\right) d\bar{S} - \frac{d}{dS}k_2(L)\right)\frac{\sin\theta(S)}{\sin\theta(L)} - \int_0^S g(\bar{S})\cos\left(\theta(\bar{S}) - \theta(S)\right) d\bar{S} \qquad \text{(EQ 114)}$$

As special cases we report certain closed form solutions for the first inverse problem. If the final bending curvature is given by

$$k_2 = k_0 \cos\left(\pi\frac{S}{2L}\right) \qquad \text{(EQ 115)}$$

the required distributed bending moment to produce the above curvature from the initial straight configuration is found to be

$$\hat{m}_2 = \frac{k_0\pi}{2L}\sin\left(\pi\frac{S}{2L}\right) \tag{EQ 116}$$

A parabolic curvature

$$k_2 = k_0\left(1 - \frac{S^2}{L^2}\right) \tag{EQ 117}$$

requires a linear distributed bending moment given by

$$\hat{m}_2 = \frac{2k_0 S}{L^2} \tag{EQ 118}$$

and a hyperbolic curvature

$$k_2 = k_0 \operatorname{sech}\left(\frac{\lambda S}{L}\right) \tag{EQ 119}$$

requires a distributed bending moment given by

$$\hat{m}_2 = \frac{\lambda k_0}{L}\operatorname{sech}\left(\frac{\lambda S}{L}\right)\tanh\left(\frac{\lambda S}{L}\right) \tag{EQ 120}$$

In the above equations k_0 is a constant measuring the amplitude of the required curvature. The value of the parameter λ measures the spread of the curvature, which a pproaches a delta function as λ becomes large.

A particular case of the first inverse problem arises, when the deformed configuration of the rod is required to be straight, i.e., $k_2=0$. Eqs. (64-66) reduce in this case to

$$\hat{m}_2 = \frac{dK_2}{dS} \tag{EQ 121}$$

The above equation suggests that the integral of the distributed bending moment is the required initial curvature to produce a straight deformed configuration.

To solve the second inverse problem we introduce the new dependent variable $\Lambda = \frac{d}{dS}(k_2 - K_2)$ and eq. (107) takes the form

$$\frac{d}{dS}\left(\frac{d\Lambda}{dS}/k_2\right) + k_2\Lambda = -\left(\frac{d}{dS}\left(\frac{d\hat{m}_2}{dS}/k_2\right) + k_2\hat{m}_2\right) \qquad \text{(EQ 122)}$$

The above equation has the same form with eq. (109) and can be solved in exactly the same way with eq. (109) of the first inverse problem.

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